2014

Geodesic Circulant Graphs Embedded on the Flat Torus

Cameron Richer
Rhode Island College, cricher_8002@email.ric.edu

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Embedded on the
Flat Torus

By

Cameron Richer

An Honors Project Submitted in Partial Fulfillment

of the Requirements for Honors

in

The Department of Mathematics

The School of Arts and Sciences

Rhode Island College

2014

Geodesic Circulant Graphs

Embedded on the

Flat Torus

An Undergraduate Honors Project Presented

By

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To

The Department of Mathematics

Approved:

Project Advisor	5/12/14 Date
Chair, Department Honors Committee	5/12/2014 Date
Department Chair	5/12/14 Date

GEODESIC CIRCULANT GRAPHS EMBEDDED ON THE FLAT TORUS

CAMERON RICHER

ABSTRACT. In this paper, we will investigate graphs that arise from the intersections of geodesics embedded on the flat torus. For any size collection of geodesics, the number of unique intersections is countable via their slopes. As well, any embedding of two geodesics gives rise to a circulant graph for which its chromatic number can be calculated from their respective slopes. Furthermore, the previously described circulant graphs embedded on the flat torus are self-dual. This provides an effective face coloring of any graph arising from the embedding of two slopes on the torus.

1. INTRODUCTION

The flat torus can be obtained from the polygonal region corresponding to the 1x1 unit square with opposing edges identified in the same directions (See Figure 1). It should be observed that through a tiling of 1x1 squares, where all orientations agree with $\mathbb{R}^2$, it is possible to generate all of the space $\mathbb{R}^2$ (See Figure 2). Therefore, $\mathbb{R}^2$ is a covering space (in fact, the universal covering space) for the flat torus. Thus, through the quotienting of $\mathbb{R}^2$ by $\mathbb{Z}^2$, one can obtain the space $(\frac{\mathbb{R}^2}{\mathbb{Z}^2}, +)$ which is isomorphic to the aforementioned unit square. The construction of the torus from its polygonal region can be seen in Figure 3. In this paper, we will investigate graphs embedded on the flat torus which arise from intersections of closed geodesics.

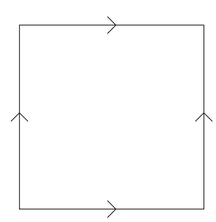


FIGURE 1. The Unit Square with opposing edges identified

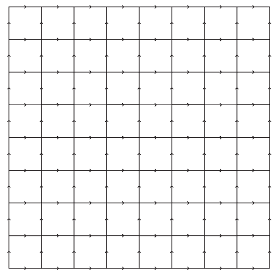


FIGURE 2. A tiling of the 1x1 Unit Square with edges identified

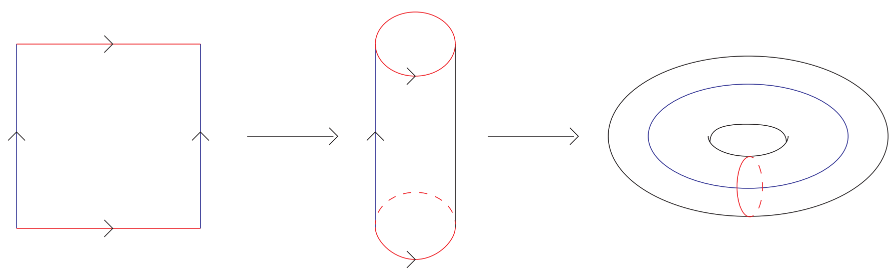


FIGURE 3. Building the Torus

For the time being, a slope is a closed geodesic beginning and ending at the point $(0,0)$ on a flat torus. We define these geodesics by the equation $\frac{p}{q}x \equiv y(\text{mod } 1)$. For any relatively prime integers p and q, a slope on the polygonal region corresponds to the equation $\frac{p}{q}x \equiv y(\text{mod } 1)$. An example of a $\frac{1}{4}$ slope can be seen in Figure 4. It should be noted that if p and q are not relatively prime, the slope would intersect itself. Also, if the slopes were represented by an irrational number, the geodesic embedded on the flat torus would not be closed (See Lemma 1.1).

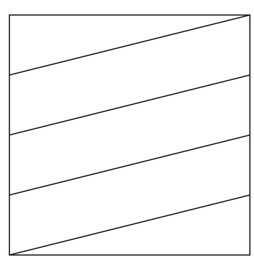


FIGURE 4. Slope $\frac{1}{4}$ embedded on the flat torus

Theorem. *Let L be a closed geodesic embedded on the flat torus. Given L has a slope of $\frac{p}{q}$, where p and q are relatively prime, L is guaranteed to hit every p^{th} partition of the form $(\frac{n}{p}, 0)$ (mod 1) on the top and bottom of the polygonal region and every q^{th} partition of the form $(0, \frac{m}{q})$ (mod 1) on the left and right sides of the polygonal region where m and n are arbitrary integers.*

Proof. Let L be a closed geodesic embedded on the flat torus. Assume L has a slope of $\frac{p}{q}$, where p and q are relatively prime integers. Realizing the flat torus as the space $\frac{\mathbb{R}^2}{\mathbb{Z}^2}$, one can note that every point (x, y) in $\mathbb{R}^2$ will lie on the top and bottom of the identified polygonal region if $y \in \mathbb{Z}$ and (x, y) will lie on the left and right boundaries of the polygonal region if $x \in \mathbb{Z}$. Therefore, for any integer n and slope with the equation $\frac{p}{q}x \equiv y(\text{mod } 1)$, the slope is guaranteed to touch the points $(\frac{np}{q}, 0)(\text{mod } 1)$ on the poligional region. Also, by analyzing $\frac{q}{p}y \equiv x(\text{mod } 1)$ for all $y \in \mathbb{Z}$, we can guarantee that the slope touches the points $(\frac{mq}{p}, 0)(\text{mod } 1)$ on the poligional region where m is any integer. □

Lemma 1.1. *Since p and q are relatively prime, $\frac{np}{q}(\text{mod } 1) \equiv \frac{n}{q}(\text{mod } 1)$ and $\frac{mq}{p}(\text{mod } 1) \equiv \frac{m}{p}(\text{mod } 1)$ for all integers m and n. Thus, there exists an isomorphism from the set of intersections the geodesic generates with the left\right borders and top\bottom borders of the flat torus and the groups $\mathbb{Z}_q$ and $\mathbb{Z}_p$ respectively.*

The fiber over a $\frac{p}{q}$ slope in the cover $\mathbb{R}^2$ is the infinite collection of lines $y = \frac{p}{q}x - \frac{n}{q}$ for any integer n (See Figure 5). A second way to observe this is as a tiling of 1x1 unit squares covering $\mathbb{R}^2$. Each tori contain disjoint line segments, arising from a slope, that when tiled create continuous lines of the form $y = \frac{p}{q}x - \frac{n}{q}$ for any integer n. The construction of the torus with a slope of $\frac{1}{2}$ can be seen in Figure 6.

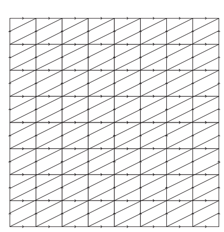


FIGURE 5. The fiber over a $\frac{1}{2}$ slope

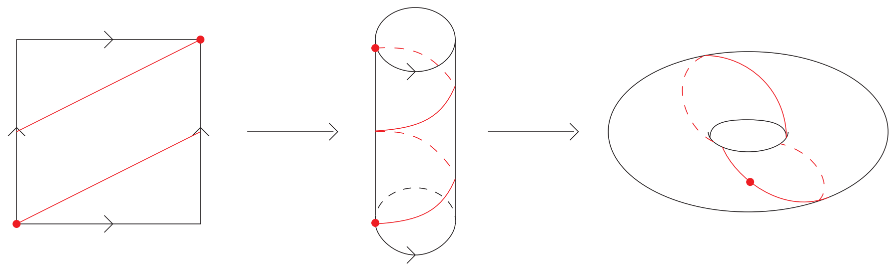


FIGURE 6. Slope $\frac{1}{2}$ on the Torus

2. Intersection of Two Geodesics

Assume now two loops are embedded on the flat torus (See Figure 7).

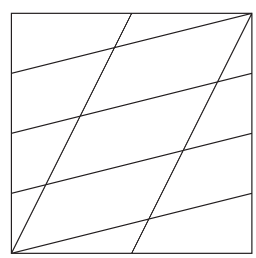


FIGURE 7. Two slopes $\{\frac{1}{4}, \frac{2}{1}\}$ embedded on the Flat Torus

Theorem. *Assume a, b, p, q are integers, where $gcd(a, b) = 1$ and $gcd(p, q) = 1$. Let L_1 and L_2 be two slopes defined by $\frac{p}{q}$ and $\frac{a}{b}$ respectively. The x values of all intersections on the torus are $x \equiv \frac{nb}{pb-aq}(mod\ b)$.*

Proof. Assume now, for integers a, b, p, q where $\gcd(a, b) = 1$ and $\gcd(p, q) = 1$, that two geodesics $\{L_1, L_2\}$ are defined by the slopes $\{\frac{p}{q}, \frac{a}{b}\}$. On the flat torus, these slopes are represented by the equations $L_1 : \frac{p}{q}x \equiv y(\bmod 1)$ and $L_2 : \frac{a}{b}x \equiv y(\bmod 1)$. To study the loops however, it is advantageous to look at the fiber over L_1 and one lift of L_2 in the covering space $\mathbb{R}^2$. Thus we obtain the equations $L_1 : \frac{p}{q}x - \frac{n}{q} = y$ for all real x and n in the integers and $L_2 : \frac{a}{b}(x) = y$ where for any integer m, x is between m and $m+b$ (For a simple case, let $m = 0$). Now, having two equations and two unknowns, the system reduces to $\frac{p}{q}x - \frac{n}{q} = \frac{a}{b}x$. Thus, $\frac{p}{q}x - \frac{a}{b}x = \frac{n}{q}$ and $(\frac{p}{q} - \frac{a}{b})x = \frac{n}{q}$. Therefore, $(\frac{pb-aq}{bq})x = \frac{n}{q}$ and $x = (\frac{bq}{pb-aq})\frac{n}{q}$. Finally, the x values of all intersections can be defined by $x = \frac{nb}{pb-aq}$ where x is between m and $m+b$. Thus, the x values of all intersections on the torus becomes $x \equiv \frac{nb}{pb-aq} (\text{mod } b)$. Also, by similar methods, all y values take the form $y \equiv \frac{na}{pb-aq} (\text{mod } a)$.

$\square$

An interesting case occurs when the two slopes take the form of $\frac{p}{q}$ and $-\frac{p}{q}$. When this specific case is present, a unique method for determining the points of intersection on the torus can be used.

Corollary 2.1. *Let L_1 and L_2 be two geodesics embedded on the flat torus defined by the slopes of $\frac{p}{q}$ and $-\frac{p}{q}$ where $p, q \in \mathbb{Z}$. Let $a \in \mathbb{Z}_{2p}$ and $b \in \mathbb{Z}_{2q}$. If $a + b \equiv 0 (mod\ 2)$, then $(\frac{a}{2p}, \frac{b}{2q})$ is an intersection of L_1 and L_2.*

Proof. Let L_1 and L_2 be two geodesics embedded on the flat torus defined by the slopes of $\pm\frac{p}{q}$ where $p, q \in \mathbb{Z}$. So $\frac{p}{q}x \equiv y (\text{mod } 1)$ and $\frac{-p}{q}x \equiv y (\text{mod } 1)$. Thus $px \equiv qy (\text{mod } 1)$ and $px \equiv qy (\text{mod } 1)$. Setting the lines equal we then obtain the system:

$$\begin{cases} -px + qy = n \text{ for } n \in \mathbb{Z} \\ px + qy = l \text{ for } l \in \mathbb{Z} \end{cases}$$

Thus we can see:

$$\begin{cases} 2px = a \text{ for } a = (l - n) \in \mathbb{Z} \\ 2qy = b \text{ for } b = (n + l) \in \mathbb{Z} \end{cases}$$

Finally, on the poligional region, we get:

$$\begin{cases} x \equiv \frac{a}{2p} (\text{mod } 1) \cong \mathbb{Z}_{2p} \\ y \equiv \frac{b}{2q} (\text{mod } 1) \cong \mathbb{Z}_{2q} \end{cases}$$

$$q\frac{a}{2p} + p\frac{b}{2q} = l$$

$$\frac{a}{2} + \frac{b}{2} = l$$

$$k + p = 2l$$

$$k + p \equiv 0 (\text{mod } 2)$$

$\square$

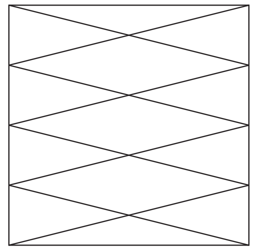
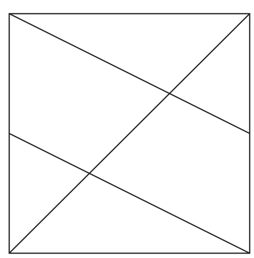


FIGURE 8. Two slopes $\{\frac{1}{4}, -\frac{1}{4}\}$ embedded on the Flat Torus

Theorem. *Assume a, b, p, q are integers, where $gcd(a, b) = 1$ and $gcd(p, q) = 1$. Let L_1 and L_2 be two slopes $\frac{p}{q}$ and $\frac{a}{b}$. The number of intersections between L_1 and L_2 is $|pb - aq|$.*

Proof. Assume a, b, p, q are integers, where $\gcd(a, b) = 1$ and $\gcd(p, q) = 1$. Let L_1 and L_2 be two loops defined by the slopes $\frac{p}{q}$ and $\frac{a}{b}$ respectively. From Theorem 2, the x values of all intersections on the torus becomes $x \equiv \frac{nb}{pb-aq} (\text{mod } b)$ such that n runs through all integers. Knowing x is between 0 and b ($0 < x \leq b$ or $b < x \leq 0$ if b is positive or negative respectively), n can be restricted to integers 1 through $pb - aq$ or $pb - aq$ through -1 if $pb - aq$ is positive or negative respectively. Thus, the number of intersections is equal to the magnitude of $pb - aq$, or $|pb - aq|$.

$\square$



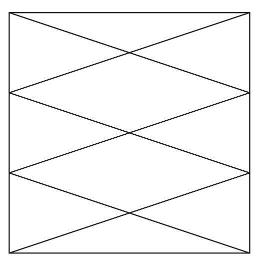
Example. Two geodesics $\{L_1, L_2\}$ embedded on the torus are represented by their slops $\{\frac{1}{1}, -\frac{1}{2}\}$.

$|pb - aq| = |(1 * 2) - (-1 * 1)| = |2 + 1| = 3$. Thus, the number of unique intersections of these slopes is 3.

Corollary 2.2. *Let L_1 and L_2 be two loops embedded on the flat torus defined by the slopes of $\pm\frac{p}{q}$ where $p, q \in \mathbb{Z}$. The number of unique intersections between L_1 and L_2 is equal to $2pq$.*

Proof. Let L_1 and L_2 be two loops embedded on the flat torus defined by the slopes of $\pm\frac{p}{q}$ where $p, q \in \mathbb{Z}$. Let $a \in \mathbb{Z}_{2p}$ and $b \in \mathbb{Z}_{2q}$. From *theorem 2*, notice that if a is even, b is also even. Also, if a is odd, b is also odd. Since the orders of $\mathbb{Z}_{2p}$ and $\mathbb{Z}_{2q}$ will always be even, exactly half of their elements will be even and the other half odd. Thus all $2p$ elements of $\mathbb{Z}_{2p}$ match with all $2q$ elements of $\mathbb{Z}_{2q}$ exactly half of the time. Therefore the total number of unique vertices is $\frac{1}{2}(2p2q) = 2pq$. Stated differently, there are p even elements and p odd elements in $\mathbb{Z}_{2p}$ as well as q even and q odd elements in $\mathbb{Z}_{2q}$. Since all even elements pair and all odd elements do the same, we get $pq + pq$ or $2pq$ total unique intersections between L_1 and L_2.

□



Example. Two loops $\{L_1, L_2\}$ embedded on the torus are represented by their slops $\{\frac{1}{3}, -\frac{1}{3}\}$.

$|pb - aq| = |(1 * 3) - (-1 * 3)| = |3 + 3| = 6$

Since $L_2 = -L_1$ we can use the corollary: $2pq = 2 * 1 * 3 = 6$

Thus, the number of unique intersections of this mesh graph is 6.

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3. Intersection of n Geodesics

Suppose now that two closed geodesics are embedded on the flat torus. The addition of a third slope through the origin presents a new issue. Not only will there be intersections between two slopes, but there will be instances where all three will intersect at a shared point. The following theorem presents a tool that will assist in counting intersections of three slopes.

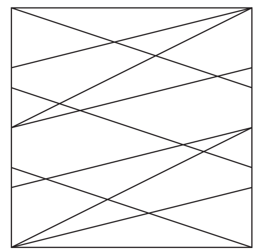


FIGURE 9. Three slopes $\{\frac{1}{4}, \frac{1}{2}, -\frac{1}{3}\}$ embedded on the Flat Torus

Theorem. *Let $a, b, x, y \in \mathbb{Z}$. The number of times $\frac{x}{a}$ equals $\frac{y}{b}$ on the interval $(0, 1]$ is equal to the greatest common divisor of (a, b).*

Proof. Let $a, b, x, y \in \mathbb{Z}$. Show that the number of times $\frac{x}{a}$ equals $\frac{y}{b}$ on the interval $(0, 1]$ is equal to the greatest common divisor of (a, b). Assume $\frac{x}{a} = \frac{y}{b}$. Since $\frac{x}{a} = \frac{y}{b}$, we know $ay = bx$, $y = \frac{b}{a}x$ and $x = \frac{a}{b}y$. Because $\frac{x}{a}$ and $\frac{y}{b}$ are on the interval $(0, 1]$, we know $0 < x \leq a$ and $0 < y \leq b$. By substitution we get $0 < \frac{b}{a}x \leq b$ and $0 < \frac{a}{b}y \leq a$. Thus, $0 < bx \leq ab$ and $0 < ay \leq ab$. Notice the first time $ay = bx$ is at the least common multiple of a and b. By definition, $ay = bx$ at each multiple of $[a, b]$ up to the highest possible multiple which, with the restrictions $0 < x \leq a$ and $0 < y \leq b$, is equal to ab. By defintion the number of multiples of $[a, b]$ from $[a, b]$ to ab is the greatest common divisor of (a, b). Thus, the number of times $\frac{x}{a}$ equals $\frac{y}{b}$ on the interval $(0, 1]$ is also equal to the greatest common divisor of (a, b).

□

Corollary 3.1. *Let $a, b, q, x, y \in \mathbb{Z}$. The number of times $\frac{qx}{a}$ equals $\frac{qy}{b}$ on the interval $(0, 1]$ is equal to the greatest common divisor of (a, b).*

Proof. Let $a, b, q, x, y \in \mathbb{Z}$. Show that the number of times $\frac{qx}{a}$ equals $\frac{qy}{b}$ on the interval $(0, q]$ is equal to the greatest common divisor of (a, b). Assume $\frac{qx}{a} = \frac{qy}{b}$. Because $\frac{qx}{a} = \frac{qy}{b}$ on the interval $(0, q]$, $0 < x \leq a$ and $0 < y \leq b$. Also, since $\frac{qx}{a} = \frac{qy}{b}$, we know $aqy = bqx$. Thus, $ay = bx$. Through the application of theorem 1, we see the number of times $\frac{x}{a}$ equals $\frac{y}{b}$ on the interval $(0, 1]$ is equal to the greatest common divisor of (a, b).

$\square$

Nevertheless, there is no limit to the number of slopes that can be embedded on the flat torus. Thus, for n slopes, we can use this same method to assist in counting the intersection of 3 or more slopes.

Theorem. *Let all x_i, a_i, and q be integers for $1 \leq i \leq n$. The number of times the fractions $\{\frac{qx_1}{a_1}, \frac{qx_2}{a_2}, \ldots, \frac{qx_n}{a_n}\}$ are equal on the interval $(0, 1]$ is equal to the greatest common divisor of $(a_1, a_2, \ldots, a_n)$.*

Proof. Let all x_i, a_i, and q be integers. Show the number of times the fractions $\{\frac{qx_1}{a_1}, \frac{qx_2}{a_2}, \ldots, \frac{qx_n}{a_n}\}$ are equal on the interval $(0, 1]$ is equal to the greatest common divisor of $(a_1, a_2, \ldots, a_n)$. Since each $\frac{qx_i}{a_i}$ is on the interval $(0, 1]$, we know $0 < x_i \leq a_i$. Assume $\frac{qx_1}{a_1} = \frac{qx_2}{a_2} = \ldots = \frac{qx_n}{a_n}$. Then $(a_2 * a_3 \ldots a_n)qx_1 = (a_1 * a_3 * a_4 \ldots a_n)qx_2 = \ldots = (a_1 * a_2 \ldots a_{n-1})qx_n$. The first time this equality will hold true is LCM$[a_2a_3 \ldots a_n, a_1a_3a_4 \ldots a_n, a_1a_2 \ldots a_{n-1}]$ while the final time is $a_1a_2 \ldots a_n$. We know:

$$\begin{aligned} &[a_2a_3 \ldots a_n, a_1a_3a_4 \ldots a_n, a_1a_2 \ldots a_{n-1}] \\ &= \frac{a_1^{n-1}a_2^{n-1} \ldots a_n^{n-1}}{(a_1^{n-1}a_2^{n-2} \ldots a_n^{n-2}, a_1^{n-2}a_2^{n-1} \ldots a_n^{n-2}, \ldots a_1^{n-2}a_2^{n-2} \ldots a_n^{n-1})} \\ &= \frac{a_1a_2 \ldots a_n}{(a_1, a_2, \ldots, a_n)} \frac{(a_1^{n-2}a_2^{n-2} \ldots a_n^{n-2})}{(a_1^{n-2}a_2^{n-2} \ldots a_n^{n-2})} \\ &= \frac{a_1a_2 \ldots a_n}{(a_1, a_2, \ldots, a_n)} \end{aligned}$$

By defintion the number of multiples of $[a_1, a_2, \ldots, a_n]$ from $[a_1, a_2, \ldots, a_n]$ to $a_1a_2 \ldots a_n$ is the greatest common divisor of $(a_1, a_2, \ldots, a_n)$. Thus, the number of times $\{\frac{qx_1}{a_1}, \frac{qx_2}{a_2}, \ldots, \frac{qx_n}{a_n}\}$ are equal on the interval $(0, 1]$ is also equal to the greatest common divisor of $(a_1, a_2, \ldots, a_n)$.

$\square$

The problem of counting the intersections between n slopes embedded on the flat torus is rooted heavily in combinatorics. To begin, Theorem 3 should be applied to each pair of slopes in the set $\{L_1, L_2, ..., L_n\}$ such that each L_i is represented by its slope $\frac{p_i}{q_i}$. For n slopes, there will be $\binom{n}{2}$ unique pairings. The sum of these numbers represents the total number of intersections, or double points, shared between all $\binom{n}{2}$ pairs of slopes. Therefore, if the same point lies on the intersections between three or more slopes, it has been overcounted a predictable number of times.

Thus, in order to count each point only one time, it is necessary to then subtract the overcounted triple points. The problem remains however: how may times have the triple points been counted? The problem can be reworded to: if three slopes intersect at a common point, how many ways can this be represented by two slopes? Stating the question in this form does all but say the answer. The solution is as simple as $\binom{3}{2}$ or 3. In order to count each triple point only once, it is required to total all double points and subtract two times the number of triple points.

$$\text{Unique Intersections} = (\sum_{i=1}^{n-1} \sum_{k=i+1}^{n} |p_i q_j - p_j q_i|) - (2 \times \text{Triple Points})$$

Nevertheless, for more than three loops, triple points are also counted as quadruple points. Thus the number of quadruple points must be subrtacted from the total triple points. In a similar fashion as above, we must ask: if four loops intersect at a common point, how many ways can this be represented by three loops? Again, $\binom{4}{3}$ or 4 ways. Therefore, we must subtract three times the number of quadruple points from triple points.

$$\text{Unique Intersections} = (\sum_{i=1}^{n-1} \sum_{k=i+1}^{n} |p_i q_j - p_j q_i|) - (2 \times \sum_{i=1}^{n-2} \sum_{i<k_1<k_2}^{n} (|p_i q_{k_1} - p_{k_1} q_i|, |p_i q_{k_2} - p_{k_2} q_i|)$$
$$-3 \times \text{Quadruple Points})$$

Furthermore, quadruple points are also counted as quintuple points. Thus the number of quintuple points must be subrtacted from the total quadruple points. Since the quintuple points are counted $\binom{5}{4}$ or 5 ways as quadruple points, we must subtract four times the total quintuple points from the total quadruple points.

$$\text{Unique Intersections} = (\sum_{i=1}^{n-1} \sum_{k=i+1}^{n} |p_i q_j - p_j q_i|) - (2 \times \sum_{i=1}^{n-2} \sum_{i<k_1<k_2}^{n} (|p_i q_{k_1} - p_{k_1} q_i|, |p_i q_{k_2} - p_{k_2} q_i|)$$
$$-(3 \times \sum_{i=1}^{n-2} \sum_{i<k_1<k_2<k_3}^{n} (|p_i q_{k_1} - p_{k_1} q_i|, |p_i q_{k_2} - p_{k_2} q_i|, |p_i q_{k_3} - p_{k_3} q_i|) - 4 \times \text{Quintuple Points}))$$

This process continues until n-tuple points have been correctly adjusted. As seen each stage forms a predictable pattern. Double points count triple points 3 times, triple points count quadruple points 4 times, quadruple points count quintuple points 5 times, and so on. In general, for a positive integer m, m-tuple points, where $m \leq n$, are counted $\binom{m}{m-1}$ ways by the $(m-1)$-tuple points. Since $\binom{m}{m-1} = \binom{m}{1} = m$, we will always subtract $(m-1)$ times the total number of m-tuple points from the total number of $(m-1)$-tuple points.

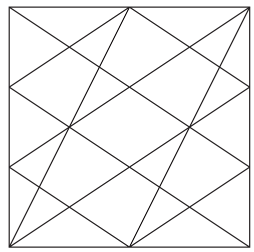
For a collection of n slopes, any grouping of m slopes, $m \leq n$, meeting at a point correspond to m-tuple points are counted in the following way as a collection of j slopes:

$$\binom{m}{j} = \text{ number of times the } m\text{-tuple intersection is represented by the intersection of } j \text{ slopes}$$

Example. In an embedding of four geodesics, quadruple points are represented in a variety of ways. That is, by:

$$\binom{4}{0} = 1 \text{ Space}$$
$$\binom{4}{1} = 4 \text{ Loops}$$
$$\binom{4}{2} = 6 \text{ Double Points}$$
$$\binom{4}{3} = 4 \text{ Triple Points}$$
$$\binom{4}{4} = 1 \text{ Quadruple Point}$$

Applying a different counting technique to justify the number of unique intersections, we see quadruple points have been added in $\binom{4}{2} = 6$ times as double points and subracted out $2 * \binom{4}{3} = 2 * 4 = 8$ times as triple points. Therefore, since quadruple points have been counted $6 - 8 = -2$ times in total, they must be added back in three times to be counted only once. Thus, we have arrived at the same conclusion.



Example. Three loops $\{L_1, L_2, L_3\}$ embedded on the torus are represented by their slopes $\{\frac{2}{1}, \frac{2}{3}, -\frac{2}{3}\}$.

$|p_1q_2 - p_2q_1| = |(2 \times 3) - (2 \times 1)| = |6 - 2| = |4| = 4$

$|p_1q_3 - p_3q_1| = |(2 \times 3) - (-2 \times 1)| = |6 + 2| = |8| = 8$

$|p_2q_3 - p_3q_2| = |(2 \times 3) - (-2 \times 3)| = |6 + 6| = |12| = 12$

$(|p_1q_2 - p_2q_1|, |p_1q_3 - p_3q_1|) = (4, 8) = 4$

$|p_1q_2 - p_2q_1| + |p_1q_3 - p_3q_1| + |p_2q_3 - p_3q_2| - 2 \times [(|p_1q_2 - p_2q_1|, |p_1q_3 - p_3q_1|)]=$

$4 + 8 + 12 - (2 \times 4) = 24 - 8 = 16$

Thus, the total number of unique intersections is 16.

Theorem. *If n slopes are embedded on the flat torus, then the number of unique vertices is:*

$$\sum_{x=1}^{n-1}[((-1)^{x+1})(x)(\sum_{i=1}^{n-x}\sum_{i<k_1<k_2<\ldots<k_x}^{n}(|p_iq_{k_1}-p_{k_1}q_i|, |p_iq_{k_2}-p_{k_2}q_i|, \ldots, |p_iq_{k_x}-p_{k_x}q_i|))]$$

The simpliest case of this formula is that of two slopes.

Example. Assume there are two geodesics embedded on the torus. Thus $n = 2$ and the two geodesics $\{L_1, L_2\}$ are represented by their slopes $\{\frac{p_1}{q_1}, \frac{p_2}{q_2}\}$.

The number of unique intersections is equal to:

$$\sum_{x=1}^{n-1}[((-1)^{x+1})(x)(\sum_{i=1}^{n-x}\sum_{i<k_1<k_2<\ldots<k_x}^{n}(|p_iq_{k_1}-p_{k_1}q_i|,|p_iq_{k_2}-p_{k_2}q_i|,\ldots,|p_iq_{k_x}-p_{k_x}q_i|))] =$$

$$\sum_{x=1}^{1}[((-1)^{x+1})(x)(\sum_{i=1}^{2-x}\sum_{i<k_1<k_2<\ldots<k_x}^{2}(|p_iq_{k_1}-p_{k_1}q_i|,|p_iq_{k_2}-p_{k_2}q_i|,\ldots,|p_iq_{k_x}-p_{k_x}q_i|))] =$$

$$\sum_{i=1}^{1}\sum_{i<k_1}^{2}(|p_iq_{k_1}-p_{k_1}q_i|) = (|p_1q_2-p_2q_1|) = |p_1q_2-p_2q_1|$$

In general, if n geodesics all intersect at a point (r, s), there exists the isometry $f((x, y)) = (x - r, y - s)$ which preserves the geometry of the torus; in particular, the intersections of the slopes. Therefore, if n geodesics have one point in common, then the methods of this paper can be applied by considering just the slopes of the geodesics. The following figure (Figure 10) shows how this isometry works on a specific example, both locally and globally on the flat torus.

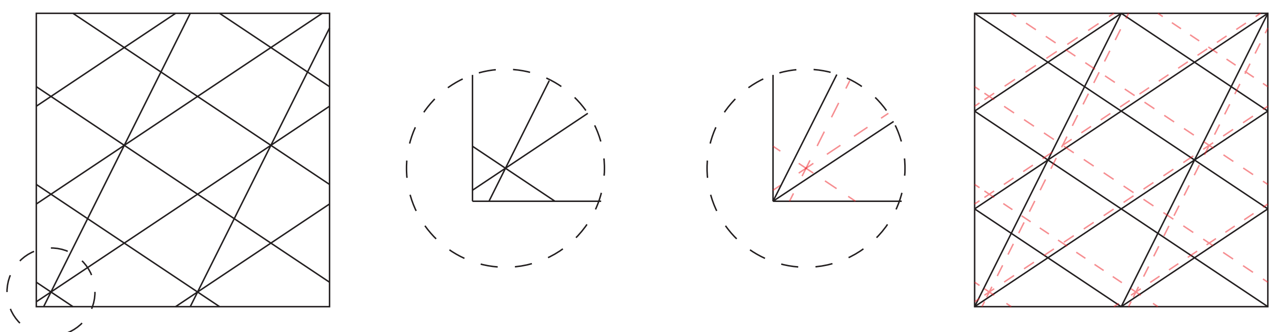


FIGURE 10. $f((x, y)) = (x - r, y - s)$ on the Flat Torus

4. Geodesic Circulant Graphs

Definition 4.1 (Mesh Graph $M_n\{L_1, L_2, \ldots, L_m\}$)**.** *For a collection of slopes $\{L_1, L_2, \ldots, L_m : L_i = \frac{p_i}{q_i}\}$ embedded on the torus, the vertices of the mesh graph are generated by the unique intersections of the slopes while the edges are the line segments of the slopes connecting the intersections.*

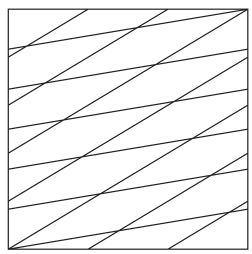
A deeper look into two loop mesh graphs yields interesting results. To begin, all mesh graphs with two loops $\{L_1, L_2\}$ represented only by their slopes $\{\frac{p_1}{q_1}, \frac{p_2}{q_2}\}$ have the formula $|p_1q_2 - p_2q_1|$ for the unique number of vertices. Locally, each vertex is the intersection of two lines. Thus every vertex of a two slope mesh graph has degree four. Traversing L_1, that is letting a real number x start at 0 and run through q_1 in the equation $y \equiv \frac{p_1}{q_1}x(\text{mod } 1)$, one arrives fairly quickly at the first vertex of the graph. Doing the same for the second loop L_2, x starts at 0 and runs through q_2 in the equation $y \equiv \frac{p_2}{q_2}x(\text{mod } 1)$. Of interest, is when traversing L_2 will land on the same first vertex that was hit by L_1. We know vertices take on the coordinates $(x_1, y_1) \equiv (\frac{nq_1}{pb-aq}, \frac{np_1}{pb-aq})(\text{mod } 1)$ or $(x_2, y_2) \equiv (\frac{nq_2}{pb-aq}, \frac{np_2}{pb-aq})(\text{mod } 1)$ on the flat torus. Thus, assuming we are located on the first vertex encountered by traversing L_1, we want to find what value of n will solve the following: $(\frac{q_1}{pb-aq}, \frac{p_1}{pb-aq}) \equiv (\frac{nq_2}{pb-aq}, \frac{np_2}{pb-aq})(\text{mod } 1)$. Thus, for integers λ_1, λ_2, we need to solve the following two congruencies: $\lambda_1 q_2 \equiv q_1(\text{mod } |p_1q_2 - p_2q_1|)$ and $\lambda_2 p_2 \equiv p_1(\text{mod } |p_1q_2 - p_2q_1|)$. Since these must be solved at the same time, we will let $\lambda = [\lambda_1, \lambda_2]$. Therefore, by replacing λ_1 and λ_2 by their least common multiple we get $\lambda q_2 \equiv q_1(\text{mod } |p_1q_2 - p_2q_1|)$ and $\lambda p_2 \equiv p_1(\text{mod } |p_1q_2 - p_2q_1|)$. By solving $\lambda(p_2 + q_2) \equiv p_1 + q_1(\text{mod } |p_1q_2 - p_2q_1|)$ for λ, we find the number of vertices traversed by L_2 to arrive at the first vertice traversed by L_1. This process can be continued at any other point reached by L_1. In other words, how many vertices does it take traversing L_2 to reach the k^{th} vertice traversed by L_1? The answer: λk. We arrive at this conclusion through the equation $(\frac{kq_1}{pb-aq}, \frac{kp_1}{pb-aq}) \equiv (\frac{knq_2}{pb-aq}, \frac{knp_2}{pb-aq})(\text{mod } 1)$ for any integer k which becomes equivalent to solving $\lambda k(p_2 + q_2) \equiv k(p_1 + q_1)(\text{mod } |p_1q_2 - p_2q_1|)$. Thus, we can see the problem then reduces once again to $\lambda(p_2 + q_2) \equiv p_1 + q_1(\text{mod } |p_1q_2 - p_2q_1|)$.

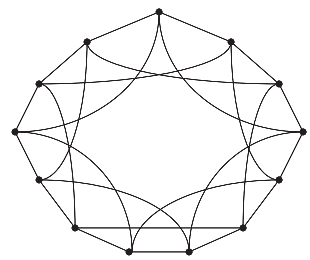
Analyzing graphs with similar properties, one important classification arose: *circulant graphs.*

Definition 4.2 (Circulant Graph, $C_n(a, b)$)**.** *A circulant graph of two jumps, $C_n(a, b)$ such that $a \neq \pm b$, is a graph of n vertices in which the i^{th} vertex is adjacent to the $(i + a)^{th}$, $(i - a)^{th}$, $(i + b)^{th}$ and $(i - b)^{th}$ graph vertices.*

The jumps of a circulant graph (a, b) can be viewed in a variety of ways. For any two slope mesh graph, there is an isomorphism to a two jump circulant graph $C_n(1, \lambda)$. The 1-jump is found by traversing one closed geodesic on the flat torus. The second jump, λ, is found through solving the aforementiond equation: $\lambda(p_2 + q_2) \equiv$

$p_1 + q_1 (\text{mod } |p_1 q_2 - p_2 q_1|)$. It is well known that $\lambda = -(n - \lambda)$ as a jump of the circulant graph. The additive inverse of λ is a jump of the same magnitude in the opposite direction.





Example. Two slopes $\{L_1, L_2\}$ embedded on the torus are represented by their slopes $\{\frac{3}{5}, \frac{1}{6}\}$.

The number of unique vertices is equal to: $|p_1 q_2 - p_2 q_1| = |(3 * 6) - (1 * 5)| = |18 - 5| = |13| = 13$.

The first jump is equal to 1. The second jump, λ is found by solving $p_1 + q_1 \equiv \lambda(p_2 + q_2)(\text{mod } |p_1 q_2 - p_2 q_1|)$. So $8 = 7\lambda(\text{mod } 13)$ implies $\lambda = 3$

Therefore, the mesh graph represented by the slopes $\{\frac{3}{5}, \frac{1}{6}\}$ is isomorphic to $C_{13}(1, 3)$.

The following theorem describes how a circulant graph of n verticies and jumps of 1 and λ can be embedded on the torus as a two slope mesh graph.

Theorem. *Any circulant graph $C_n(1, \lambda)$ such that $(\lambda, n) = 1$, can be embedded as a two sloped mesh graph on the torus.*

Proof. We want to find two slopes $\frac{a}{b}$ and $\frac{c}{d}$ such that $(a, b) = 1$, $(c, d) = 1$ and the resulting circulant graph is $C_n(1, \lambda)$. We can let $a = 1$ and $b = \lambda - 1$, notice $(a, b) = 1$.

So we want a c and d such that the system of equations $c + d = 1$ and $-bc + ad = n$ is satisfied. Thus:

$$\begin{bmatrix} 1 & 1 \\ -b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ n \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{a+b} \begin{bmatrix} a & -1 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 \\ n \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{a+b} \begin{bmatrix} a-n \\ b+n \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} a-n \\ b+n \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} 1-n \\ \lambda-1+n \end{bmatrix}$$

Let c and d be defined as above. Since c and d are integers and $c+d=1$, then $d=1-c$ and $c=1-d$. Therefore, $d \equiv 1(\text{mod } c)$ and $c \equiv 1(\text{mod } d)$. Thus, $(c,d)=1$.

□

Since every two loop mesh graph is a circulant graph of the form $C_n(1,\lambda)$, properties known for circulant graphs can be used to study mesh graphs. Several studies have been conducted about circulant graphs that have answered the question of graph coloring. The following formula details the chromatic number of all two jump circulant graphs of the form $C_n(a,b)$ such that $a \neq \pm b$.

Theorem.

$$\chi(C_n(a,b)) = \begin{cases} 2 & \text{if } n \text{ even, } a, b \text{ odd} \\ 5 & \text{if } n=5 \\ 4 & \text{if } n=13,\ a=1,\ b=5 \\ 4 & \text{if } n \neq 5,\ n (\text{mod } 3) \neq 0,\ a=1,\ b \in \{2, \frac{n-1}{2}\} \\ 3 & \text{otherwise} \end{cases} \qquad [1]$$

In order to find the chromatic number of two slope mesh graphs embedded on the torus, the above formula can be simplified to classify all circulant graphs of the form $C_n(1,\lambda')$ such that $(\lambda',n)=1$. We define $\lambda' = \min(\lambda, n-\lambda)$

Theorem.

$$\chi(C_n(1,\lambda')) = \begin{cases} 2 & \textit{if } n \textit{ even} \\ 5 & \textit{if } n=5,\ \lambda' \neq 1 \textit{ or } 4 \textit{ (mod } 5) \\ 4 & \textit{if } n=13,\ \lambda'=5 \\ 4 & \textit{if } n \neq 5,\ n \textit{(mod 3)} \neq 0,\ \lambda' \in \{2, \frac{n-1}{2}\} \\ 3 & \textit{otherwise} \end{cases}$$

Proof. The following cases are derived from the theorem detailing the chromatic number of $C_n(a,b)$. In these cases however, $a=1$ and $b=\lambda'$ where the only restriction on λ' is that $(\lambda', n) = 1$. Thus, we allow the case that $\lambda' = \pm 1$.

<u>Case 1</u>: n even

With n even, no even λ' will generate a Mesh Graph. Any odd λ' will yield a chromatic number of 2.

<u>Case 2</u>: $n = 5$

The previous theorem disallowed a $\lambda' = 1$ or 4. When these cases occur, we obtain a chromatic number of 3. For all other λ' however $(2,3)$ (mod 5), the chromatic number is 5.

<u>Case 3</u>: $n = 13$, $\lambda' = 5$

This specific case follows from the previous theorem.

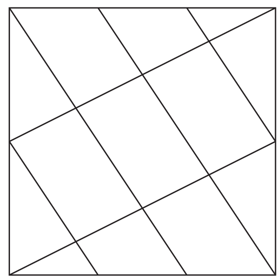
<u>Case 4</u>: $n \neq 5$, n(mod 3)$\neq 0$, $\lambda' \in \{2, \frac{n-1}{2}\}$

This specific case follows from the previous theorem.

<u>Case 5</u>: n odd, $\lambda' = \pm 1$

As we now allow a λ' of ± 1, we obtain the result for this case that the chromatic number is 3. We allow this to become a part of the previous theorem's otherwise case.

$\square$



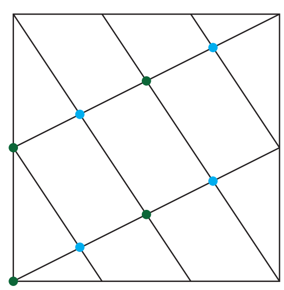
Example. Two slopes $\{L_1, L_2\}$ embedded on the torus are represented by their slopes $\{\frac{1}{2}, -\frac{3}{2}\}$.

The number of unique vertices is equal to: $|p_1q_2 - p_2q_1| = |(1*2) - (-3*2)| = |2-(-6)| = |8| = 8$.

λ is found by solving $p_1 + q_1 \equiv \lambda(p_2 + q_2)(\text{mod } |p_1q_2 - p_2q_1|)$. So $3 = -1\lambda(\text{mod } 8)$ implies $\lambda = 5$

Therefore, our mesh graph, $M_8\{\frac{1}{2}, -\frac{3}{2}\}$ is equivalent to $C_8\{1,5\}$

Since $n = 8$, n is even. Thus, $\chi(C_8\{1,5\}) = 2$



5. The Self-Dual Mesh Graph

In order to find the proper face coloring of two slope mesh graphs, we must analyze the dual graph. It is well known that line segments from midpoint to midpoint on opposite sides are parallel to the sides of the parallelogram. Thus, after constructing the the dual, one obtains a shift of the origional mesh graph.

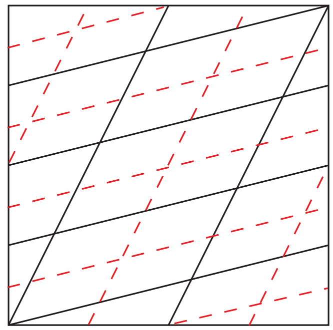


FIGURE 11. $M_7\{\frac{1}{4}, \frac{2}{1}\}$ and its dual (dotted)

By selecting one intersection of the dual and making that the arbitrary origin, we are able to generate an identical mesh graph (See Figure 10). Therefore, we can say that the dual graph of a two slope mesh graph is isomorphic to its corresponding mesh graph. Thus, the dual graph has the same vertex coloring as the original mesh graph. With this information, we can say the minimum number of colors needed to color each face of the mesh graph is equal to the chromatic number described by the above function.

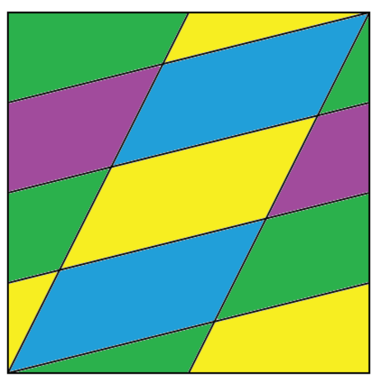


FIGURE 12. Proper face coloring of $M_7\{\frac{1}{4}, \frac{2}{1}\}$

References

[1] Sara Nicoloso, Ugo Pietropaoli. *Vertex Colouring of Circulant Graphs: A Combinatorial Approach*, Istituto Di Analisi Dei Sistemi Ed Informatica, 2007.